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METHOD OF CALCULATING THE EFFECT OF A GUST ON AN ARBITRARY THIN WING

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ABSTRACT

Development of a numerical method of calculating unsteady flow past a thin wing moving in an ideal incompressible medium with the aid of a diagram of a supporting surface. Since the time and coordinate dependences of the boundary conditions on the wing surface can be any desired, it is shown that the proposed method can be used to study aperiodic motion of the wing as a solid body, arbitrary deformations of the wing, entry of the wing into a gust, the effect of a weak shock wave on the wing, etc. The method is said to be applicable to monoplane wings of any shape in the plane, to annualr wings, to systems of similar wings, etc.

A detailed summary of articles which compute the influence of a gust on a wing is contained in the monographs (Ref. 1, 2). Without discussing this summary, we would like to point out that the effective solution of the problem is only obtained for a profile in the case of subsonic velocities.

1. <u>Main Assumptions of the Method.</u> /51*

Let us assume that an arbitrary, thin supporting surface performs unsteady motion as a solid body, or is deformed and enters gradually or instantaneously into a gust. If we know the characteristics of the gust, and the laws of motion and deformation of the body, we may readily determine the normal velocity components corresponding to them on the surface of the wing, and it may thus be assumed that they are known. The normal component of the disturbed velocity W_n, produced during the motion of the supporting surface, must compensate for them at any moment of time (condition of smooth flow). We should point out that, within the framework of the linear theory to be considered later on, the interaction of the gust, the motion of the wing as a solid body, and the deformation of its surface may be studied independently.

For purposes of definition, we shall investigate the monoplane wing of an arbitrary, but symmetrical, form in a plane. Let us introduce a moving coordinate system Oxyz, which is connected with the wing, with the origin at the middle of the root chord b (Figure 1). We shall assume that the main velocity (which does not depend on the time t) of the wing U_0 **is parallel** to the root chord (0x axis).

In the general case, the boundary conditions at the wing may be represented as follows

$$\frac{W_v}{U_0} = cf\left(\frac{x_0}{b}; \frac{z_0}{b}', \tau\right) \qquad (1.1)$$
$$\tau = \frac{U_0 t}{b}$$

* Note: Numbers in the margin indicate pagination in the original foreign text.

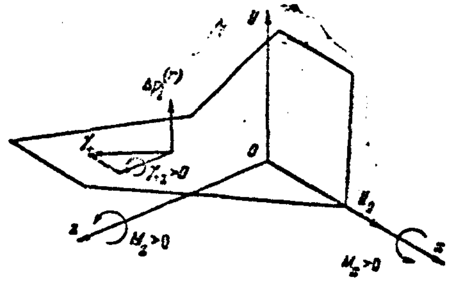

Figure 1

Here x_0, z_0 are coordinates of a point on the wing surface; τ -- dimensionless time; and c -- a certain normalization, dimensionless constant. For example, it is advantageous to select c so that $|f| \leq 1$ for /52 the confined right portions of (1.1). In the linear problem, it may be assumed that f = 0 in the case of $\tau < 0$. We should also point out that, if the solid wing moves at a zero angle of attack and gradually enters a sharply confined gust, then the function f = 0 in the wing portion which does not enter the gust.

It may be assumed that the perturbed motion of the liquid outside of the wing, and the vortical track behind it, is a potential motion. In the case of both steady and unsteady motions, the potential of the perturbed velocities will satisfy the Laplace equation. The pressure must change continuously at the vortex sheet behind the wing, and the Chaplygin-Zhukovskiy condition is satisfied at the trailing edges of the wing.

In the case of arbitrary unsteady motion, the pressure difference Δp at the lower and upper wing surfaces may be expressed by the intensity

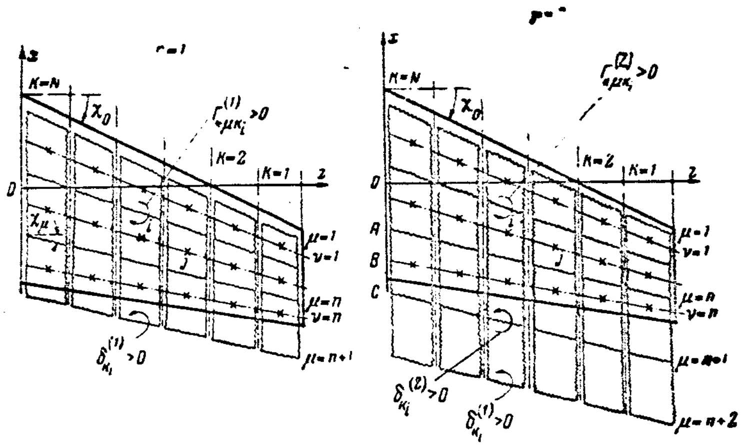

Figure 2

of the attached vortex layer γ_+ on the basis of the Zhukovskiy theorem "in a small region" (Ref. 3)

$$\Delta p = -\rho\gamma_+ W_{00n} \tag{1.2}$$

Here W_{00n} is the normal to the vortex axis γ_+ corresponding to the relative velocity of the medium at a point in the vortex layer, and ρ is the medium density.

As is known, the velocity potential corresponding to the vortex surface or to discrete vortices satisfies the Laplace equation. According to (1.2), if the normal component of the relative medium velocity equals zero, then there is no pressure difference on the free surface. Consequently, the axes of the free vortices must be directed over the local velocity of the medium, or these vortices must move at a velocity equal to it. Within the framework of the linear theory, it may be assumed that this velocity equals U_0. Instead of (1.2), we shall then have

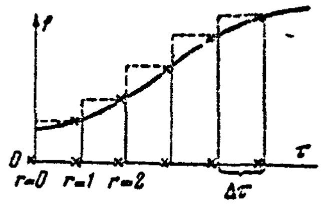

Figure 3

$$\Delta p = \rho \gamma_{+z} U_0 \tag{1.3}$$

The intensity of the attached vortices, whose axes are perpendicular to the velocity U_0, is designated by γ_{+z}. The positive directions Δp and γ_{+z} are shown in Figure 1. Thus, if the wing is replaced by a vortex layer, several conditions of the problem are automatically fulfilled. It is only necessary to select the intensity of this layer in such a way that the conditions of smooth flow and of Chaplygin-Zhukovskiy are fulfilled.

The basic concept of the numerical method for solving this problem, /53
which will be discussed below, consists of changing from continuous distributions and processes to discrete ones.

In the first place, the continuously distributed vortex layer, which replaces the wing, is approximately modeled by a system of discrete vortices. However, in contrast to the procedure in (Ref. 3, 4), this substitution is performed here not for the attached vortex layer, but for the total layer consisting of free and attached vortices on the wing and free vortices behind the wing (Figure 2).

In the second place, the continuous process of changing the boundary condition and circulation with time is replaced by a step-wise process.

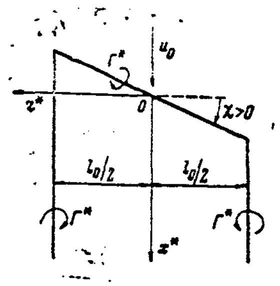

Figure 4

At specific moments of time, there is a jump-like change in the boundary conditions and circulations, and the circulations do not change in the intervals between them. Corresponding to this, the free vortices converge with the attached vortices at discrete moments in time (Figure 3).

2. <u>Vortex model of the wing</u>.

For monoplane wings, oblique, horseshoe vortices will be the main elementary vortex systems (Figure 4). The circulations of the transverse vortex and the longitudinal (parallel to the velocity U_0) semi-infinite vortex are constant and equal Γ^*. According to (Ref. 3), the velocity produced by such a vortex system in the x^*, z^*, plane may be represented in the following form

$$W_y = \frac{U_0\Gamma}{2\pi} w_y(\xi_0, \zeta_0, \chi), \quad \Gamma^* = U_0 l_0 \Gamma, \quad \xi_0 = \frac{2x_0^*}{l_0}, \quad \zeta_0 = \frac{2z_0^*}{l_0} \tag{2.1}$$

The dimensionless function, whose expression is given in (Ref. 3), is designated by w_y (ξ_0, ζ_0, χ).

The transition from the continuous change in circulation with time to a discrete change makes it possible to regard the indicated stationary vortex as the basic elementary system. Let us assume that the circulation

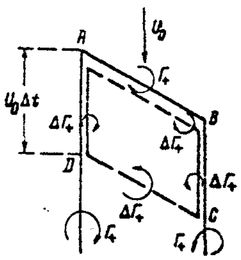

Figure 5

of the attached vortex Γ^* on an arbitrary wing element changes in a certain period of time t by the quantity $\Delta\Gamma_+$. This will be accompanied by the descent of the transverse free vortex of circulation $\Delta\Gamma_+$. At the time $t + \Delta t$, the vortex system shown in Figure 5 is obtained, instead of the oblique stationary vortex (Figure 4).

We should point out that the closed vortex filament ABCD of constant circulation $\Delta\Gamma_+$ may be regarded as the sum of two transverse vortices AB and CD of equally opposed circulation with the semi-infinite longitudinal vortices corresponding to them.

It is advantageous to replace the continuous vortex layer by discrete vortices directly on the wing just as was done in the stationary case (Ref. 3) and in the problem of harmonic wing oscillations (Ref. 4).

The wavy lines in Figure 2 indicate vortices, and the crosses /54
indicate the computed points at which the boundary conditions are satisfied. We shall designate the number of transverse vortices by i, the computed points -- j; vortex filaments -- μ; computed lines -- ν; and cross sections parallel to the Ox axis -- k. The significant factor

in this arrangement is firstly that the computed points lie in the middle between the vortices closest to them. In the second case, in each band of k the last computed point lies closer to the tail than the last vortex.

Generally speaking, the selection of the requisite amount of vortices on the wing $m = nN$ and the time intervals $\Delta\tau$ between the computed moments is done independently. When the circulation is computed, it is not advantageous to separate the vortices into free and attached vortices. Thus, in contrast to the cases indicated above, a total layer, which includes attached and free vortices, and not an attached layer, is simulated here by the transverse vortices on the wing at each moment of time.

Let us discuss in greater detail the transition from the changes in the boundary condition which are continuous in time to a step-wise change (Figure 3). We shall take the times indicated by crosses as the computed moments of time, and we shall characterize them by the number r

$$\tau_r = r\Delta\tau \quad (r = 0, 1, 2, \ldots) \tag{2.2}$$

They are chosen so that each of them directly precedes the moments at which there is a jump-like change in the boundary condition, and consequently in the circulation of vortices on the wing. These circulations are only computed at the time τ_r, with allowance for the fact that -- during the change from the preceding computed moment to the subsequent computed moment -- the free vortices behind the wing are carried away below the flow at the distance $U_0 \Delta t$.

At each computed moment of time, which is characterized by r, the entire vortex layer on the wing is replaced by a system of oblique vortices, as is shown in Figure 2. A system of free vortices behind the wing is also produced by means of the oblique vortices whose position with respect to the wing has not been fixed, and changes for different r. The sweepback angle of the free vortices behind the wing will be the same as for the last vortex filament on the wing $\mu = n$. Figure 2 shows the vortex systems of the wing in the case of $r = 1$, when one free vortex filament $\mu = n + 1$ is formed behind the wing, and in the case of $r = 2$, when there are two such filaments ($\mu = n + 1$ and $\mu = n + 2$).

The positions of the transverse free vortices behind the wing must be selected so that the Chaplygin-Zhukovskiy condition is fulfilled at the trailing edges of the wing. This condition causes the intensity of the attached vortex layer γ_+ at the trailing edges to vanish.

In order to fulfill this condition, it is necessary that the free vortices behind the wing are located at a large distance from the last computed line $\nu = n$ at the computed moments of time. As computations have shown, the Chaplygin-Zhukovskiy condition is fulfilled if we set AB = BC (Figure 2). This means that we may assume that the minimum distance between the free vortex filament $\mu = n + 1$ and the computed line $\nu = n$ is equal to the maximum distance between the vortex line $\mu = n$ and the line $\nu = n$.

The wing chord at each cross section is separated into n equal sections. The distance between the adjacent vortex filaments (Ref. 3) at each cross section k will equal b_k/n, and therefore AB = = $\frac{1}{2}b/n$.

In the time Δt the transverse free vortex passes through the distance AC = $U_0 \Delta t$. Assuming that AB = BC = $\frac{1}{2}$AC, we find

$$\Delta\tau = 1/n, \qquad \Delta\tau = U_0\Delta t/b \tag{2.3}$$

According to (1.1) and (2.2), we obtain

$$\tau_r = \frac{r}{n}, \qquad \tau_r = \frac{t_r U_0}{b} \qquad (r = 1, 2, \ldots) \tag{2.4}$$

where t_r and τ_r are the dimensional and dimensionless computed times.

3. <u>Calculation of the vortex positions and computed points</u>.

The circulation of each vortex will depend on the computed time, which can be characterized by the number r for a given number n of vortex filaments on the wing according to (2.4), and the number of the vortex i. It is more advantageous to define the vortex position on the wing by two quantities: the number of the filament μ and the number of the band k_i, which corresponds to the vortex i. Assuming that the span l_0 of each transverse vortex is the same, we shall investigate the dimensional and dimensionless circulations of the total, attached and free vortices on the wing. /55

$$\Gamma_{\Sigma\mu k_i}^{(r)} = U_0 l_0 \Gamma_{*\mu k_i}^{(r)}, \qquad \Gamma_{+\mu k_i}^{(r)} = U_0 l_0 \Gamma_{\mu k_i}^{(r)}, \qquad \Gamma_{-\mu k_i}^{(r)} = U_0 l_0 \Delta_{\mu k_i}^{(r)} \tag{3.1}$$
$$(1 \leqslant \mu \leqslant n, \quad 1 \leqslant k_i \leqslant N)$$

The circulations of free vortices which descend from the wing do not change with time. Therefore, it can be assumed that they are dependent either on μ and k_i or on r and k_i

$$\Gamma_{k_i}^{(r)} = U_0 l_0 \delta_{k_i}^{(r)}, \qquad (n+1 < \mu \leqslant n+r, \quad 1 \leqslant k_i \leqslant N) \tag{3.2}$$

N is the number of bands into which the semi-span of the wing is divided in formulas (3.1) and (3.2).

Numbering the transverse vortices i and the computed points j from right to left, beginning with $\mu = 1$ and $\nu = 1$ (Figure 2), we can write the following for the wing of any form in a plane

$$\begin{aligned} i &= k_i + (\mu - 1) N \qquad (i = 1, \ldots, m; \quad 1 \leqslant \mu \leqslant n) \\ j &= k_j + (\nu - 1) N \qquad (j = 1, \ldots, m; \quad 1 \leqslant \nu \leqslant n) \end{aligned} \qquad (m = nN) \tag{3.3}$$

Let us derive the relationship for determining the requisite geometric parameters of a wing with constant sweepback over the leading and trailing edges (a wing of such a type is shown in Figure 2). Let l be the span; b -- the root chord; b_k -- the chord at the cross section k; λ -- the wing aspect ratio; η -- contraction; χ_0 -- sweepback angle over the leading edge of the wing; and χ_μ -- sweepback angle of the filament μ. We then have (Ref. 3)

$$\begin{gathered} \frac{l}{b} = \frac{\lambda}{2}\frac{\eta+1}{\eta}, \qquad \operatorname{tg}\chi_\mu = \operatorname{tg}\chi_0 - \frac{4\Delta_\mu}{\lambda}\frac{\eta-1}{\eta+1} \qquad \left(\lambda = \frac{l^2}{S}, \quad \eta = \frac{b}{b_k}\right) \\ \Delta_\mu = \frac{\mu}{n} - \frac{3}{4n}, \qquad \frac{b_k}{b} = 1 - z_k^* \frac{\eta - 1}{\eta}, \quad z_k^* = \frac{2z_k}{l}, \qquad \frac{l_0}{l} = \frac{1}{2N} \end{gathered} \tag{3.4}$$

Let us set

$$(x_\mu^{k_i}, \ z_{k_i}), \qquad (x_{0\nu}^{k_j}, \ z_{0k_j})$$

-- the coordinates of the middle of the transverse vortex i and the computed point j, respectively. We then have (Ref. 3)

$$z_k^* = 1 - \frac{2k-1}{2N}, \qquad \frac{x_\mu^{k_i}}{b_{k_i}} = \frac{1}{2} - \frac{4\mu-3}{4n} + \frac{\Delta x_{k_i}}{b_{k_i}}$$

$$\frac{x_{0\nu}^{k_j}}{b_{k_j}} = \frac{1}{2} - \frac{4\nu-1}{4n} + \frac{\Delta x_{k_j}}{b_{k_j}}, \qquad \frac{\Delta x_k}{b_k} = -\frac{b}{b_k} z_k^* \left(\frac{\lambda}{4}\frac{\eta+1}{\eta}\operatorname{tg}\chi_0 - \frac{\eta-1}{2\eta}\right) \tag{3.5}$$

Either k_i or k_j may be substituted in the formulas, where k occurs without an index.

Let us turn to vortex filaments behind the wing $\mu = n + \varepsilon$, $\varepsilon = 1, \ldots, r$, which are obtained by the parallel shift of the filament $\mu = n$ along the Ox axis by $\varepsilon b/n$. Retaining the same notation as was used for the wing, we would like to note that all the relationships given above which do not include μ remain unchanged. In addition, we have

$$\chi_{n+\varepsilon} = \chi_n \qquad (\varepsilon = 1, \ldots, r) \tag{3.6}$$

In accordance with these considerations, the following relationship holds (Figure 2)

$$x_{n+\varepsilon}^{k_i} = x_n^{k_i} - \varepsilon b/n$$

Let us find the expression for the dimensionless coordinates in which the velocities (2.1) are computed at the point j, calculated both from the vortex i, which is located on the same half of the wing, and from the vortex i' which is symmetrical with i. Comparing the coordinate systems of Figures 2 and 4, after the obvious transformations, we shall have /56

$$\xi_{0\mu\nu}^{k_ik_j} = 4\frac{b}{l}N\left(\frac{b_{k_i}}{b}\frac{x_\mu^{k_i}}{b_{k_i}} - \frac{b_{k_j}}{b}\frac{x_{0\nu}^{k_j}}{b_{k_j}}\right)$$

$$\zeta_0^{k_ik_j} = 2N(z_{k_i}^* - z_{0k_j}^*), \qquad (\zeta_0 + \delta\zeta_0)^{k_ik_j} = 2N(z_{k_i}^* + z_{k_j}^*) \tag{3.7}$$

When the velocities from the free vortices behind the wing are computed, the dimensionless coordinates $\zeta_0^{k_ik_j}$ and $(\zeta_0 + \delta\zeta_0)\, k_ik_j$ will be determined by the same formulas. According to (3.6), we can write

$$\xi_{0n+\varepsilon}^{k_i k_j} = \frac{2}{l}\left(x_{n+\varepsilon}{}^{k_i} - x_{0\nu}{}^{k_j}\right)$$

We thus have

$$\xi_{0n+\varepsilon\nu}{}^{k_i k_j} = \xi_{0n\nu}^{k_i k_j} - \frac{8N\varepsilon}{n\lambda}\frac{\eta}{\eta+1} \qquad (\varepsilon = 1, \ldots, r) \tag{3.8}$$

The dimensionless velocities (2.1), which correspond to the vortices i and i' and are computed by these arguments, can be designated by

$$w_{\nu\mu\nu}^{k_i k_j} = w_\nu(\xi_{0\mu\nu}^{k_i k_j},\ \zeta_0{}^{k_i k_j},\ \chi_\mu), \quad \delta w_{\nu\mu\nu}^{k_i k_j} = w_\nu(\xi_{0\mu\nu}^{k_i k_j},\quad (\zeta_0 + \delta\zeta_0)^{k_i k_j},\ \chi_\mu) \tag{3.9}$$

and we thus have

$$\chi_{n+\varepsilon} = \chi_n \qquad (\varepsilon = 1, \ldots, r) \tag{3.10}$$

4. <u>Equations for circulation</u>.

Let us compile the equations for determining the dimensionless circulations of total vortices on the wing $\Gamma_{*\mu k_i}^{(r)}$ and the free vortices behind the wing $\delta_{k_i}^{(r)}$. At each computed moment τ all of the total circulations on the wing must be determined anew, and only the circulations on the line $\mu = n + 1$ are unknown for the free vortices. Thus, for any r, the number of the unknowns equals m + N (Figure 2).

The superscript for $\delta_{k_i}^{(r)}$ indicates that for any value of r, the given free vortices occur on the line $\mu = n + 1$. In addition, these vortices converge below the flow, and their circulation does not change. Therefore, in the case of $r = \varepsilon$, the following relationship is established between the number of the line behind the wing μ and the circulation of free vortices δ on it:

$$\mu = n + 1, \quad n + 2, \quad \ldots, \quad n + \varepsilon$$
$$\delta = \delta_{k_i}^{(\varepsilon)}, \quad \delta_{k_i}^{(\varepsilon-1)}, \quad \ldots, \quad \delta_{k_i}^{(1)}$$

On the basis of (2.1) and Figures 2 and 4, we can write the following for the velocity produced by the entire vortex system of the wing:

$$\begin{aligned} \frac{W_y}{U_0} = \frac{1}{2\pi} \sum_{k_i=1}^{N} \sum_{\mu=1}^{n} \Gamma^{(r)}_{\circ\mu k_i} \left(w^{k_i k_j}_{y\mu\nu} \pm \delta w^{k_i k_j}_{y\mu\nu}\right) - \\ - \frac{1}{2\pi} \sum_{k_i=1}^{N} \sum_{\varepsilon=1}^{r} \delta_{k_i}^{(\varepsilon)} \left(w^{k_i k_j}_{y n+r-\varepsilon+1\nu} \pm \delta w^{k_i k_j}_{y n+r-\varepsilon+1\nu}\right) \end{aligned} \tag{4.1}$$

The positive directions for the total and free vortices are shown in Figure 2.

The plus sign in these formulas is chosen for symmetrical circulations, and the minus sign is chosen for circulations which are antisymmetric with respect to the $z = 0$ plane.

In order to determine them, we have the conditions regarding smooth flow around the wing (1.1) and the circulation of all the vortices on the wing and behind it equalling zero at each cross section k_i = const for any r.

The boundary condition will be satisfied at the points j for the computed times τ_r. Therefore, the right part of the equation will contain the functions

$$f_{k\nu}^{(r)} = f\left(x_{0\nu}^{k_j}/b, \; z_{0k_j}/b, \; \tau_r\right) \tag{4.2}$$

Taking these considerations into account, on the basis of (1.1), (4.1) and (4.2), we obtain the following systems of algebraic linear equations: /57

$$
\begin{aligned}
&\frac{1}{2\pi}\sum_{k_i=1}^{N}\sum_{\mu=1}^{n}\frac{\Gamma_{*\mu k_i}^{(r)}}{c}\left(w_{\nu\mu\nu}^{k_i k_j} \pm \delta w_{\nu\mu\nu}^{k_i k_j}\right) - \\
&- \frac{1}{2\pi}\sum_{k_i=1}^{N}\sum_{\varepsilon=1}^{r}\frac{\delta_{k_i}^{(\varepsilon)}}{c}\left(w_{\nu n+r-\varepsilon+1\nu}^{k_i k_j} \pm \delta w_{\nu n+r-\varepsilon+1\nu}^{k_i k_j}\right) = f_{k_j\nu}^{(r)} \\
&\sum_{\mu=1}^{n}\frac{\Gamma_{*\mu k_i}^{(r)}}{c} - \sum_{\varepsilon=1}^{r}\frac{\delta_{k_i}^{(\varepsilon)}}{c} = 0 \quad (r = 1, \ldots, m = nN;\ k_i = 1, \ldots, N \\
&k_j = 1, \ldots, N;\ \nu = 1, \ldots, n)
\end{aligned}
\tag{4.3}
$$

For each value of r, the solution of these systems is performed independently, beginning with r = 1, and so on, for consecutively increasing r. The unknowns will be $\Gamma_{*\mu k_i}^{(r)}/c$ and $\delta_{k_i}^{(r)}/c$, since the quantities $\delta_{k_i}^{(\varepsilon)}/c$ will already have been found in the case of $\varepsilon < r$.

5. Determination of attached vortex circulations.

According to the Zhukovskiy theorem "in the small region" (1.3), the aerodynamic loads of the wing may be expressed by the circulation of attached vortices. Therefore, we must find the circulation of the attached vortices $\Gamma_{\mu k_i}$ from the total circulation, found above, on the wing $\Gamma_{*\mu k_i}^{(r)}$ at each computed time τ_r.

In order to do this, in each cross section k it is advantageous to change from oblique vortices to straight vortices having the same circulation (Figure 6). It follows from (1.3) that these vortex systems produce the same normal forces.

Let us examine the arbitrary cross section k_i (Figure 6). In this cross section, the distance between the adjacent vortices will be the same and will equal b_{k_i}/n. Let us assume that Δt_{k_i} is the time during which a free vortex, which drifts at the velocity U_0, passes through the

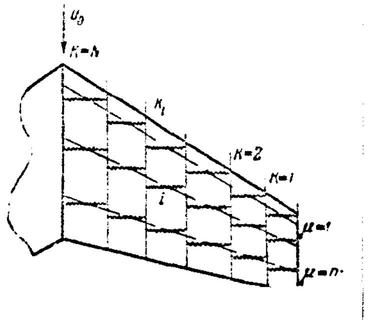

Figure 6

indicated distance. Similarly to (1.1) and (2.4), let us introduce the dimensionless quantities

$$\tau_{r_{k_i}} = r_{k_i} \Delta\tau_{k_i} \qquad (r_{k_i} = 1, 2, \ldots) \tag{5.1}$$

and let us designate $\tau_{r_{k_i}}$ as the computed moments of the cross section k_i.

On the one hand, $\Delta\tau_{k_i} = \Delta t_{k_i} U_0/b$, and on the other hand $\Delta t_{k_i} U_0 = b_{k_i}/n$. We then have

$$\Delta\tau_{k_i} = \frac{1}{n}\frac{b_{k_i}}{b} \tag{5.2}$$

As a result, we have a relationship, which is similar to (2.4) and which enables us to find the computed moments at each cross section k_i

$$\tau_{r_{k_i}} = \frac{r_{k_i}}{n}\frac{b_{k_i}}{b} \tag{5.3}$$

By analogy with (3.1), the dimensionless circulations of total, attached, and free vortices on the wing at the times $\tau_{r_{k_i}}$ will be designated by $\Gamma_{*\mu k_i}^{(r_{k_i})}$, $\Gamma_{\mu k_i}^{(r_{k_i})}$, $\Delta_{\mu k_i}^{(r_{k_i})}$. Let us take the direction shown in Figure 2 for the total circulations as the positive direction for the first two vortices, and for the free vortices it is natural to take the inverse direction as the positive direction, just as for $\delta_k^{(r)}$. Based /58 on the data in Section 4, we can readily compute the values of $\Gamma_{*\mu k_i}^{(r_{k_i})}$ by interpolation. For this purpose, for the value of r_{k_i} under consideration, it is necessary to find those r for which

$$\tau_r \leqslant \tau_{r_{k_i}} \leqslant \tau_{r+1}, \quad \text{or} \quad r \leqslant r_{k_i} b_{k_i}/b \leqslant r+1$$

hold.

The latter is obtained according to (2.4) and (5.3).

Thus, the selection of the values of r which are requisite for interpolation is done for given r_{k_i} on the basis of the following inequality

$$0 \leqslant r_{k_i} b_{k_i}/b - r \leqslant 1 \tag{5.4}$$

After this, if -- for example -- we confine ourselves to linear interpolation, we may write

$$\Gamma_{*\mu k_i}^{(r_{k_i})} = \Gamma_{*\mu k_i} + (\Gamma_{*\mu k_i}^{(r+1)} - \Gamma_{*\mu k_i}^{(r)}) \left(r_{k_i} \frac{b_{k_i}}{b} - r\right) \tag{5.5}$$

Let us compute the circulation of the attached vortices $\Gamma_{\mu k_i}^{(r_{k_i})}$, assuming that the total circulations $\Gamma_{*\mu k_i}^{(r_{k_i})}$ are known. Since the circulation of the attached vortex changes due to the descent of the free vortex, we then have

$$\Gamma_{\mu k_i}^{(r_{k_i})} - \Gamma_{\mu k_i}^{(r_{k_i}-1)} = \Delta_{\mu k_i}^{(r_{k_i})} \tag{5.6}$$

The free vortices drift below along the flow, and the computed times τ_{k_i} are selected so that during the time between them the vortices traverse exactly the distance between the adjacent lines μ and $\mu + 1$

(Figures 3 and 6).

Therefore, at the computed times τ_{k_i} there will never be free vortices on the filament $\mu = 1$; on the second filament, there will only be vortices from the first, on the third filament there will only be vortices from the second in the case of $r_{k_i} = 1$; in the case of $r_{k_i} = 2$ -- from the second and third, etc. Taking these considerations into account, we may write

$$\begin{gathered} \Gamma_{*1k_i}^{(r_{k_i})} = \Gamma_{1k_i}^{(r_{k_i})}, \qquad \Gamma_{*2k_i}^{(r_{k_i})} = \Gamma_{2k_i}^{(r_{k_i})} - \Delta_{1k_i}^{(r_{k_i})} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Gamma_{*\mu k_i}^{(r_{k_i})} = \Gamma_{\mu k_i}^{(r_{k_i})} - \Delta_{\mu-1k_i}^{(r_{k_i})} - \Delta_{\mu-2k_i}^{(r_{k_i}-1)} - \cdots - \Delta_{\mu-r_{k_i}k_i}^{(1)} \end{gathered} \tag{5.7}$$

On the basis of (5.6) and (5.7), we may obtain the recurrence formula for computing the circulation of attached vortices

$$\Gamma_{\mu k_i}^{(r_{k_i})} = \Gamma_{*\mu k_i}^{(r_{k_i})} + \left[\Gamma_{\mu-1k_i}^{(r_{k_i})} - \Gamma_{\mu-1k_i}^{(r_{k_i}-1)}\right] + \left[\Gamma_{\mu-2k_i}^{(r_{k_i}-1)} - \Gamma_{\mu-2k_i}^{(r_{k_i}-2)}\right] + \cdots + \Gamma_{\mu-r_{k_i}k_i}^{(1)} \tag{5.8}$$

It may be assumed that all circulations with non-positive indices equal zero in formulas (5.6) - (5.8).

6. Computation of wing aerodynamic characteristics.

First of all, by interpolation over time we must change from $\Gamma_{\mu k_i}^{(r_{k_i})}$ to $\Gamma_{\mu k_i}^{(r)}$ -- to the circulation of attached vortices at the computed times τ_r which are the same for every wing. In order to do this, we must also find $\tau_{r_{k_i}}$, when the following inequality is fulfilled for given τ_r

$$\tau_{r_{k_i}} \leqslant \tau_r \leqslant \tau_{r_{k_i}+1}, \text{ or } 0 \leqslant rb_j/b_{k_i} - r_{k_i} \leqslant 1 \tag{6.1}$$

The latter holds according to (2.4) and (5.3).

We can determine $r_{k_i} = 1, 2, \ldots$ from (6.1) for the values of $r =$ /59
$= 1, 2, \ldots$. After this, confining ourselves to linear interpolation,

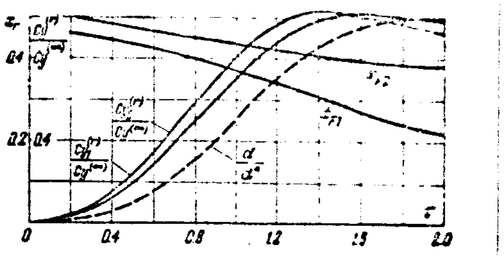

Figure 7

for example, we have

$$\Gamma_{\mu k_i}^{(r)} = \Gamma_{\mu k_i}^{(r_{k_i})} + \left[\Gamma_{\mu k_i}^{(r_{k_i}+1)} - \Gamma_{\mu k_i}^{(r_{k_i})}\right](rb/b_{k_i} - r_{k_i}) \tag{6.2}$$

On the basis of the Zhukovskiy theorem "in the small region" (1.3), we may now readily determine unsteady aerodynamic loads on the wing, the cross section characteristics, and the total coefficients of the wing.

We must take the fact into account that at each section $\Delta x = b_{k_i}/n$ the continuously distributed layer is replaced by one discrete vortex.

According to Figure 1, for the supporting force and the moments of the wing, we have expressions in which summation is performed over one (right) half of the wing

$$\begin{gathered} Y^{(r)} = 2\sum_{i=1}^{m} \Delta p_i^{(r)} \Delta S_i \\ M_z^{(r)} = 2\sum_{i=1}^{m} \Delta p_i^{(r)} x_i \Delta S_i \\ M_x^{(r)} = -2\sum_{i=1}^{m} \Delta p_i^{(r)} z_i \Delta S_i \\ \Delta S_i = \frac{l_0 b_{k_i}}{n} \end{gathered} \tag{6.3}$$

Here i is the number of the attached vortex; ΔS_i -- the corresponding elementary surface; x_i, z_i -- coordinates for the middle of the vortex.

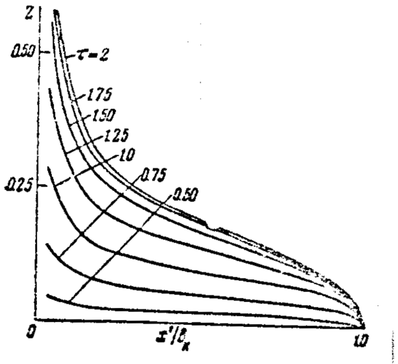

Figure 8

On the basis of (1.3), we have

$$\Delta p_i^{(r)} \Delta S_i = \rho U_0^2 l_0^2 \Gamma_{\mu k_i}^{(r)} \tag{6.4}$$

Let us introduce the aerodynamic coefficients

$$c_y^{(r)} = \frac{2Y^{(r)}}{\rho U_0^2 S}; \quad m_z^{(r)} = \frac{2M_z^{(r)}}{\rho U_0^2 S b}, \quad m_x^{(r)} = \frac{2M_x^{(r)}}{\rho U_0^2 S l} \tag{6.5}$$

where S is the wing surface. We then obtain the following from (6.3), (6.4), and (6.5)

$$\begin{aligned} c_y^{(r)} &= \frac{\lambda}{N} \sum_{k_i=1}^{N} \sum_{\mu=1}^{n} \Gamma_{\mu k_i}^{(r)} \\ m_z^{(r)} &= \frac{\lambda}{N} \sum_{k_i=1}^{N} \sum_{\mu=1}^{n} \Gamma_{\mu k_i}^{(r)} \frac{b_{k_i}}{b} \frac{x_\mu^{k_i}}{b_{k_i}} \\ m_x^{(r)} &= -\frac{\lambda}{2N} \sum_{k_i=1}^{N} \sum_{\mu=1}^{n} \Gamma_{\mu k_i}^{(r)} z_{k_i} \end{aligned} \tag{6.6}$$

Similar formulas hold for the wing cross section coefficients

$$c_{y k_i}^{(r)} = \frac{\lambda}{2N} \frac{\eta+1}{\eta} \frac{b}{b_{k_i}} \sum_{\mu=1}^{n} \Gamma_{\mu k_i}^{(r)}, \qquad m_{z k_i}^{(r)} = \frac{\lambda}{2N} \frac{\eta+1}{\eta} \frac{b}{b_{k_i}} \sum_{\mu=1}^{n} \Gamma_{\mu k_i}^{(r)} \frac{x_\mu^{k_i}}{b_{k_i}} \tag{6.7}$$

$$c_{y k_i}^{(r)} = \frac{2\Delta Y^{(r)}{}_{k_i}}{\rho U_0^2 l_0 b_{k_i}}, \qquad m_{z k_i}^{(r)} = \frac{2\Delta M^{(r)}{}_{z k_i}}{\rho U_0^2 l_0 b_{k_i}^2} \tag{6.8}$$

We shall employ $\Delta Y^{(r)}{}_{k_i}$ and $\Delta M_z^{(r)}{}_{k_i}$ to designate the supporting force /60
and longitudinal moment with respect to the Oz axis, produced in the

cross section k_i.

7. Examples.

Let us present certain data characterizing the interaction of the medium when the wing changes smoothly from a zero angle of attack to the angle $\alpha = \alpha^*$ for an equilateral triangular wing 1 and a rectangular wing 2 having the same wing aspect ratio $\lambda = 2.31$.

The law for the change in the angle of attack over dimensionless time has the following form

$$\frac{\alpha}{\alpha^*} = \frac{\tau}{2} + \frac{1}{2\pi}\sin\pi(\tau - 1), \quad (0 < \tau \leq 2), \qquad \frac{\alpha}{\alpha^*} = 1, \quad (2 \leq \tau < \infty) \tag{7.1}$$

and is shown by the dashed line in Figure 7.

Figure 8 illustrates the nature of the intensity distribution of the attached vortex layer on wing 1 in the cross section $z^* = 0.917$ located close to the end of the wing, at different moments in time, where $Z = [(\gamma + z)/\alpha^* U_0] b_k/b$. We have employed x' to designate the distance from the leading edge to a point in the cross section ($x'/b_{k_i} =$ $= 0$ -- spout and $x'/b_{k_i} = 1$ -- trailing edge of the cross section).

Figure 7 shows a change in the supporting force coefficient and a shift in the wing focus with time. The indices indicate for which wing the coefficients are chosen. The dimensionless coordinates of the focus are computed from the spout of the mean aerodynamic chord and pertain to this chord (Ref. 3). It should also be noted that the dashed curve, besides the dependence of α/α^* on τ, gives the approximate law for the change in the coefficient of supporting force emanating from the so-called hypothesis of steadiness.

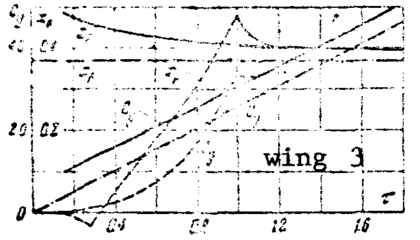

Figure 9

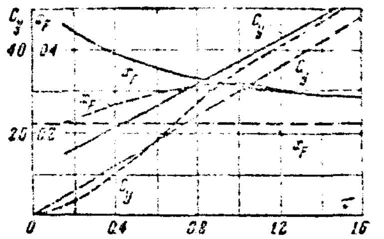

Figure 10

The influence of a vertical gust which is variable with time is examined for a triangular wing 3 and a rectangular wing 4 with the wing aspect ratio $\lambda = 2.5$. It is characterized by the following relationship

$$f = 0, \quad \tau < 0; \qquad f = \tau, \quad \tau > 0 \tag{7.2}$$

Figure 9 (wing 3) and Figure 10 (wing 4) show the manner in which the coefficient of the supporting force and the position of the focus change with time when a gust instantaneously encompasses the wing (solid lines) and when the wing gradually enters the gust (dashed line). The data obtained from the steadiness hypothesis are shown by the dot-dash lines.

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